

SEMICONTINUITY OF MEASURE THEORETIC ENTROPY FOR NONCOMPACT SYSTEMS

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ABSTRACT. We prove the upper semicontinuity of the measure theoretic entropy for the geodesic flow on complete Riemannian manifolds without focal points and bounded sectional curvature. We then study the relationship between the escape of mass phenomenon and the measure theoretic entropy on finite volume nonpositively curved manifolds satisfying the Visibility axiom. We provide a general criterion for the same relation to hold between the escape of mass and the measure theoretic entropy. This gives a criterion for the existence of measures of maximal entropy for the geodesic flow on some nonpositively curved manifolds. Finally, we prove some results in the context of countable Markov shifts.

1. INTRODUCTION

Entropy has had a prominent role in dynamical systems since its first appearance, it provides a way to measure the complexity of a dynamical system. For a complete survey we refer the reader to [Ka2]. Despite the theory of entropy has been extended to actions of a vast class of groups, i.e. amenable or sofic groups, we will restrict our attention to dynamical systems generated by a single transformation. Let us briefly recall the context on which our results sit in. Let T be a continuous endomorphism on a (not necessarily compact) metrizable space X . In the noncompact case we will always assume that the borelian σ -algebra of X is standard. We have two different flavors of entropy associated to (X, T) : its topological entropy and its measure theoretic entropy. This two quantities depends on different data. The topological entropy depends on a metric d generating the topology of X , we denote this quantity by $h_d(T)$. The measure theoretic entropy depends on a T -invariant probability measure μ , we denote this quantity by $h_\mu(T)$. We remark that if X is compact, then $h_d(T)$ is independent of the metric d . For precise definitions see Section 2. A fundamental relationship between this two quantities is the so called *variational principle*. This states that, for a locally compact metrizable space X , the following holds:

$$h_{top}(T) = \sup_{\mu \in \mathcal{M}_T} h_\mu(T).$$

Here $h_{top}(T) = \inf_d h_d(T)$, where the infimum runs over all compatible metrics on X and \mathcal{M}_T is the space of T -invariant probability measures. A natural question to ask is under what circumstances this supremum is attained. If we know the existence of measures of maximal entropy, then one can ask what special properties do they have. Both questions are far from being understood in the general setting. If we assume that X is compact and some extra hypothesis on the system, then one can answer the first question affirmatively, as the following two results show.

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Theorem 1. [R. Bowen] [Bo2] *Let (X, d) be a compact metric space and assume T is h -expansive. Then there exists a measure of maximal entropy.*

Theorem 2. [S. Newhouse] [N] *Let X be a compact smooth manifold and assume T is of class C^∞ . Then there exists a measure of maximal entropy*

In both cases what they are actually proving is the upper semicontinuity of the map $\mu \mapsto h_\mu(T)$. In the compact case this is enough to establish the existence of measure of maximal entropy. Indeed, by taking a sequence $\{\mu_n\}_{n \geq 0}$ of measures with $h_{\mu_n}(T)$ converging to $h_{\text{top}}(T)$, then any limit measure of the sequence $\{\mu_n\}_{n \geq 0}$ will have maximal entropy (the existence of the limit measure follows from the compactness of \mathcal{M}_T when X is compact). Indeed,

Theorem 3. *Assume (X, d, T) satisfies the hypothesis of Theorem 1 or Theorem 2. Let $\{\mu_n\}_{n \geq 0}$ be a sequence of invariant probability measures converging to μ . Then*

$$\limsup_{n \rightarrow \infty} h_{\mu_n}(T) \leq h_\mu(T).$$

In fact, the situations described in Theorem 1 and Theorem 2 are particular cases of *asymptotically h -expansive* systems, property which is known to imply the upper semicontinuity of the entropy map in the compact case (see [M], [Bu1]). After the work of Y. Yomdin [Y] and S. Newhouse [N] many people have investigated the relationship between the regularity of the dynamics and the upper semicontinuity of the entropy map. If T is only assumed to be C^k , the entropy map does not need to be upper semicontinuous anymore. In that case there are bounds that control the amount of failure for the entropy map to be upper semicontinuous. We refer the reader to [Bu1], [Bu2] and references therein for more details. In this paper we investigate the upper semicontinuity of the entropy map in the noncompact case. Under certain hypothesis we are able to prove that the upper semicontinuity holds at ergodic measures.

Theorem A *Let X be a topological manifold endowed with a compatible metric d . Assume $T : (X, d) \rightarrow (X, d)$ is Lipschitz. Moreover assume that (X, d, T) satisfies a simplified entropy formula. Let $\{\mu_n\}_{n \geq 0}$ be a sequence of T -invariant probability measures converging to an ergodic T -invariant probability measure μ . Then*

$$\limsup_{n \rightarrow \infty} h_{\mu_n}(T) \leq h_\mu(T).$$

For precise definitions on the hypothesis of Theorem A see Section 3. A corollary of this result will be the upper semicontinuity of the entropy map for the geodesic flow on Riemannian manifolds without focal points and bounded sectional curvature. We remark that if X is noncompact then Theorem A does not immediately implies the existence of measures of maximal entropy. This is because of the escape of mass phenomenon, \mathcal{M}_T is not necessarily compact anymore. A more detailed version of the upper semicontinuity result can be proven for the geodesic flow if we restrict ourselves to the class of nonpositively curved manifolds with the Visibility axiom. In this case we give a criterion for the existence of a measure of maximal entropy. The authors do not know of any other result concerning the existence of measures of maximal entropy in the noncompact nonpositively curved case. In contrast with the negatively curved case, where a lot of information is known (for

instance see [PPS]), in nonpositive curvature the ergodic theory of the geodesic flow is not very developed. An important result in this context is the uniqueness of the measure of maximal entropy for compact rank 1 manifolds due to Knieper [Kn]. We have the following result.

Theorem B *Let (X, g) be a complete Riemannian manifold without focal points and bounded sectional curvature. Denote by $\{g_t\}_{t \in \mathbb{R}}$ to the geodesic flow on T^1X . Let $\{\mu_n\}_{n \geq 0}$ be a sequence of g_1 -invariant probability measures converging to an ergodic g_1 -invariant probability measure μ . Then*

$$\limsup_{n \rightarrow \infty} h_{\mu_n}(g) \leq h_{\mu}(g).$$

If we moreover assume that X is nonpositively curved, that the universal cover of X satisfies the Visibility axiom and that X has finite volume, then for every convergent sequence of ergodic measures we have

$$\limsup_{n \rightarrow \infty} h_{\mu_n}(g) \leq \|\mu\| h_{\frac{\mu}{\|\mu\|}}(g) + (1 - \|\mu\|) \bar{\delta}_{\mathcal{P}}.$$

In particular, if $\bar{\delta}_{\mathcal{P}} < h_{\text{top}}(g)$, then there exists a measure of maximal entropy for the geodesic flow.

It worth mentioning that $\bar{\delta}_{\mathcal{P}}$ is a quantity that measures the complexity of the ends of X , it is the maximum of the critical exponents of parabolic subgroups of $\pi_1(X)$. At the end of Section 4 we generalize this result and provide a criterion for the existence of measure of maximal entropy, but for simplicity we do not state that result here. We finally prove an upper semicontinuity result for countable Markov shifts, which immediately extends to suspension flows over countable Markov shifts.

Theorem C *Let (Σ, σ) be a countable Markov shift and μ an ergodic Gibbs measure which is an equilibrium state for its potential. Then for every sequence $\{\mu_n\}_{n \geq 1}$ of ergodic σ -invariant probability measures converging to μ we have*

$$\limsup_{n \rightarrow \infty} h_{\mu_n}(\sigma) \leq h_{\mu}(\sigma).$$

The paper is organized as follows. In Section 2 we recall some basics facts about Ergodic Theory. In Section 3 we prove Theorem A. In Section 4 we discuss applications to the geodesic flow on nonpositively curved manifolds, we prove Theorem B and a generalization of Theorem B. In Section 5 we prove Theorem C and a corollary for suspension flows over countable Markov shifts.

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2. PRELIMINARIES

In this section (X, d) will always stand for a metrizable topological space X with a compatible metric d . We will always assume that the borelian σ -algebra of (X, d) is standard, this is immediately satisfied if X is compact. In this paper we will be interested in the dynamics of a single continuous transformation T on X . We call the triple (X, d, T) a *dynamical system*. We emphasize that in our definition we do

not a priori assume that X is compact. Define metrics d_n on X by the following formula:

$$d_n(x, y) = \max_{i \in \{0, \dots, n\}} d(T^i x, T^i y).$$

We denote by $B_n(x, r)$ to the ball centered at x of radius r in the metric d_n . A ball $B_n(x, r)$ is also called a (n, r) -dynamical ball. Given a compact subset K of X we denote by $N(K, n, r)$ to the minimum number of (n, r) -dynamical balls needed to cover K . The *topological entropy* of (X, d, T) is defined as

$$h_d(T) = \sup_{K \subset X} \lim_{r \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log N(K, n, r),$$

where the last limit runs over compact subsets of X . This definition was first introduced by R. Bowen in [Bo1] to extend the classical definition of topological entropy on compact spaces to the noncompact setting. A second way of measuring the complexity of (X, d, T) depends on a measure on X . We say that μ is a T -invariant probability measure if μ is a measure on X with total mass 1 and $\mu(T^{-1}A) = \mu(A)$ for every measurable set A . Given a countable measurable partition $\mathcal{P} = \{P_i\}_{i \in I}$ of X we define the entropy of \mathcal{P} as

$$H_\mu(\mathcal{P}) = - \sum_{i \in I} \mu(P_i) \log \mu(P_i).$$

Given two partitions \mathcal{P} and \mathcal{Q} we can construct the smallest common refinement of \mathcal{P} and \mathcal{Q} , this is denoted by $\mathcal{P} \vee \mathcal{Q}$. Define

$$h_\mu(T, \mathcal{P}) = \lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{P}\right).$$

By taking the supremum over all countable partitions of finite entropy we obtain the *entropy of T with respect to μ* , this is denoted by $h_\mu(T)$. In other words

$$h_\mu(T) = \sup_{\mathcal{P}} h_\mu(T, \mathcal{P}),$$

where the supremum runs over all countable partitions of finite entropy. The space of T -invariant probability measures on X is denoted by \mathcal{M}_T . We can assign to each element $\mu \in \mathcal{M}_T$ its entropy $h_\mu(T)$. We will refer to the mapping $\mu \mapsto h_\mu(T)$ the *entropy map*. We endow \mathcal{M}_T with the *vague topology*, this is defined by declaring that $\lim_{n \rightarrow \infty} \mu_n = \mu$ if for every continuous function of compact support f we have

$$\lim_{n \rightarrow \infty} \int f d\mu_n = \int f d\mu.$$

With this topology the space \mathcal{M}_T is not compact, sequences of measures might be losing mass. The space of T -invariant measures with total mass at most one is compact by Banach-Alaoglu theorem. The following definition will provide one of the hypothesis under which the upper semicontinuity of the entropy map can be verified.

Definition 1. A T -invariant measure μ is called a Gibbs measure with respect to a continuous map $F : X \rightarrow \mathbb{R}$ if there exists a constant $P(F) \in \mathbb{R}$ such that for every compact subset K of X there exists $r_0(K)$ for which the following holds. For $r \leq r_0(K)$ there exists $C_{K,r}$ such that

$$C_{K,r}^{-1} \leq \frac{\mu(B_n(x, r))}{e^{\sum_{k=0}^{n-1} F(T^k(x)) - nP(F)}} \leq C_{K,r},$$

whenever $x \in K \cap T^{-n}K$. The constant $P(F)$ is called the pressure of F . A measure μ is called a Gibbs measure if there exists a function F for which μ is a Gibbs measure with respect to F .

We remark that in the definition above μ is not assumed to be a probability measure. In the context of symbolic dynamics a more adequate definition of Gibbs measure is given in Section 5. The theory of Gibbs measure in the context of Markov shifts of finite type is classical and we refer the reader to [Bo3]. Many of these results have been extended to the context of Markov shifts over countable alphabet with certain finiteness assumptions (e.g. BIP condition). For precise information we refer the reader to [Sa2]. A similar theory has been developed around Gibbs measures for suspension flows over Markov shifts over finite/countable alphabet. For precise information see [BI] and references therein. For the geodesic flow on negatively curved manifold a succesful theory was recently developed in [PPS].

This two flavors of entropy seem, at first, a bit unrelated. The following theorem provides a very strong connection between them. It helps to understand the measure theoretic entropy in the spirit of the topological entropy.

Theorem 4. [A. Katok] *Let (X, d) be a compact metric space and $T : X \rightarrow X$ a continuous transformation. Let μ be an ergodic T -invariant probability measure and $\delta \in (0, 1)$. Then*

$$h_\mu(T) = \lim_{r \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{1}{n} \log N_\mu(n, r; \delta),$$

where $N_\mu(n, r; \delta)$ is the minimum number of (n, r) -dynamical balls needed to cover a set of μ -measure at least $1 - \delta$. In particular the limit above is independent of $0 < \delta < 1$.

Even though the compactness assumption is standard, we are mostly interested in the noncompact case (which is a considerably less explored territory). It will be very helpful to have a version of Katok's formula in the noncompact setting. If one follows the proof of Theorem 4 (see Section 1 in [Ka1]), it is clear that the inequality

$$h_\mu(T) \leq \lim_{r \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{1}{n} \log N_\mu(n, r; \delta),$$

holds regardless of the compactness of X . For most of our purposes this inequality will be enough. Recently F. Riquelme [R] proved the equality when X is assumed to be a manifold and T a Lipschitz map.

Theorem 5. *Let (X, d) be a metric space and $T : X \rightarrow X$ a continuous transformation. Then for every ergodic T -invariant probability measure μ we have*

$$h_\mu(T) \leq \lim_{r \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{1}{n} \log N_\mu(n, r; \delta),$$

where $N_\mu(n, r; \delta)$ as in Theorem 4. If we moreover assume (X, d) is a topological manifold with a compatible metric and T is Lipschitz, then the equality holds.

The following two definitions are inspired in standard properties of the geodesic flow on a nonpositively curved manifold with curvature bounded below. In Proposition 3 we will see a basic consequence of this properties, which is the starting point of the proof of Theorem A.

Definition 2. Let $T : (Y, \rho) \rightarrow (Y, \rho)$ be a continuous map. We say that (Y, ρ, T) has the Propagation of Closeness property or Property PC if the following is true. Given $\delta > 0$, there exist positive numbers $\epsilon(\delta)$ and $n(\delta)$ such that if $n(\delta) \leq n$, $\rho(x, y) < \epsilon(\delta)$ and $\rho(T^n x, T^n y) < \epsilon(\delta)$, then $\rho(T^k x, T^k y) < \delta$, for $k \in \{0, \dots, n\}$.

Remark 1. For all our applications we will not specify that we only consider $n \geq n(\delta)$. This is because in all the applications we will be only interested in very large n , in particular much larger than $n(\delta)$.

Definition 3. We say that a metric space (Y, ρ) has bounded geometry if for each $r > 0$ and $r' \in (0, r)$ there exists a constant $C(r, r')$ such that for any point $y \in Y$, the ball $B(y, r)$ can be covered by at most $C(r, r')$ balls of radius r' .

We finish this section with the following proposition used in [R]. Details of this proof will be needed in the proof of Theorem A.

Proposition 1. Let X be a topological manifold with compatible metric d and $T : (X, d) \rightarrow (X, d)$ a Lipschitz map. Let K be a compact subset of X of positive measure and $r > 0$. Moreover assume $\mu(\partial K) = 0$. Then there exists a countable partition \mathcal{P} of K such that $\overline{\mathcal{P}} = \{\mathcal{P}, (M \setminus K)\}$ has finite entropy with respect to any invariant probability measure μ . If $x \in K \cap T^{-n}K$, then $\overline{\mathcal{P}}^n(x) \subset B_n(x, r)$. Moreover $\mu(\partial \overline{\mathcal{P}}) = 0$.

Proof. Let $A_n = \{x \in K : T^n x \in K, T^i x \in X \setminus K \text{ for } i \in \{1, \dots, n-1\}\}$, and L the Lipschitz constant of T . By Kac's Lemma we know that for any T -invariant probability measure μ we have that $\sum_{n \geq 1} n\mu(A_n) \leq 1$, and if μ is ergodic the equality holds. Following Mañé [Ma] we can refine each A_n by at most $C(L^n/r)^d$ balls of diameter rL^{-n} and obtain a partition \mathcal{P} of K with finite entropy and $\mu(\partial \mathcal{P}) = 0$. We denote by S_k to the collection of atoms in \mathcal{P} refining A_k . Finally define $\overline{\mathcal{P}} = \{\mathcal{P}, (M \setminus K)\}$. It follows immediately from the definition of A_n and the diameter of the elements of the partition \mathcal{P} that whenever $x \in K \cap T^{-n}K$, then $\overline{\mathcal{P}}^n(x) \subset B_n(x, r)$. \square

3. PROOF OF THEOREM A

The proof of Theorem 6 is based on a simplification of Katok's entropy formula. As we will see below, the geodesic flow on nonpositively curved manifolds is the main example where this property holds.

Definition 4. We say that (X, d, T) satisfies a simplified entropy formula if for suitably chosen r and δ we have

$$h_\mu(T) = \lim_{n \rightarrow \infty} \frac{1}{n} \log N_\mu(n, r, \delta).$$

In other words Katok's entropy formula does not depends on r .

Theorem 6. Let X be a topological manifold endowed with a compatible metric d . Assume $T : (X, d) \rightarrow (X, d)$ is Lipschitz. Moreover assume that (X, d, T) satisfies a simplified entropy formula. Let $\{\mu_n\}_{n \geq 0}$ be a sequence of T -invariant probability measures converging to an ergodic T -invariant probability measure μ . Then

$$\limsup_{n \rightarrow \infty} h_{\mu_n}(T) \leq h_\mu(T).$$

Proof. Choose r and δ as in the definition of the simplified entropy formula. Let K a compact set with $\mu(K) > 1 - \delta/3$, as in Proposition 3. Denote by $\overline{\mathcal{P}}$ the partition given by Proposition 1. Define

$$B_{N,\epsilon}^\mu = \{x \in M \mid \mu(\overline{\mathcal{P}}^n(x)) > \exp(-n(h_\mu(\overline{\mathcal{P}}) + \epsilon)), \forall n \geq N\}.$$

Observe that by Shannon-McMillan-Breiman theorem $\mu(B_{N,\epsilon}^\mu) \rightarrow 1$ as $N \rightarrow \infty$. Pick N_∞ such that $\mu(B_{N_\infty,\epsilon}^\mu) > 1 - \delta/3$ and define by $J_k = K \cap T^{-k}K \cap B_{N_\infty,\epsilon}^\mu$. Observe $\mu(J_k) > 1 - \delta$. Assume $n \geq N_\infty$. By the lower bound of the measure of the elements of the partition $\overline{\mathcal{P}}^n$ on the set $B_{N_\infty,\epsilon}^\mu$ we know each J_k is covered by at most $\exp(n(h_\mu(\overline{\mathcal{P}}) + \epsilon))$ elements of $\overline{\mathcal{P}}$. We conclude that whenever $k \geq N_\infty$, J_k can be covered by $\exp(k(h_\mu(\overline{\mathcal{P}}) + \epsilon))$ (k, r) -dynamical balls. Using Proposition 3 we conclude

$$h_\mu(T) = \lim_{n \rightarrow \infty} \frac{1}{n} N_\mu(n, r; \delta) \leq h_\mu(\overline{\mathcal{P}}) + \epsilon.$$

Since ϵ is arbitrary and independent of $\overline{\mathcal{P}}$ we conclude $h_\mu(T) = h_\mu(\overline{\mathcal{P}})$. The same formula holds for every ergodic T -invariant probability measure. By ergodic decomposition it also holds for every (not necessarily ergodic) T -invariant probability measure. Recall that by construction of $\overline{\mathcal{P}}$, it satisfies $\mu(\partial \overline{\mathcal{P}}) = 0$. Since the partition $\overline{\mathcal{P}}$ is not finite we can not use the standard fact that $\eta \mapsto h_\eta(\mathcal{Q})$ is upper semicontinuous at η for every finite partition \mathcal{Q} with $\eta(\partial \mathcal{Q}) = 0$. To fix this we will use our next Lemma. To simplify notations from now on $\overline{\mathcal{P}}$ will be denoted by \mathcal{Q} .

Lemma 1. *Let \mathcal{Q} as in the paragraph above and μ an ergodic T -invariant probability measure with $\mu(\partial \mathcal{Q}) = 0$. Then the map $\eta \mapsto H_\eta(\mathcal{Q})$ is continuous at μ .*

Proof. Let μ_n be a sequence of T -invariant probability measures converging to μ . By definition of \mathcal{Q} we have

$$H_\eta(\mathcal{Q}) = - \sum_{k \geq 1} \sum_{P \in S_k} \eta(P) \log \eta(P) - \eta(X \setminus K) \log \eta(X \setminus K).$$

Observe that

$$\begin{aligned} - \sum_{k \geq M} \sum_{P \in S_k} \eta(P) \log \eta(P) &= - \sum_{k \geq M} \eta(A_k) \sum_{P \in S_k} \frac{\eta(P)}{\eta(A_k)} \log \eta(P) \\ &= - \sum_{k \geq M} \eta(A_k) \sum_{P \in S_k} \frac{\eta(P)}{\eta(A_k)} \log \frac{\eta(P)}{\eta(A_k)} - \sum_{k \geq M} \eta(A_k) \log \eta(A_k) \\ &\leq \sum_{k \geq M} \eta(A_k) \log(C(L^k/r)^d) - \sum_{k \geq M} \eta(A_k) \log \eta(A_k) \\ &\leq d \log L \sum_{k \geq M} k \eta(A_k) + \log(C/r^d) \sum_{k \geq M} \eta(A_k) - \sum_{k \geq M} \eta(A_k) \log \eta(A_k). \end{aligned}$$

By Kac's lemma we know that for every ergodic T -invariant measure η we have $\sum_{k \geq 1} k \eta(A_k) = 1$. In general we only have $\sum_{k \geq 1} k \eta(A_k) \leq 1$. Since μ is an ergodic measure we can find $M = M(\epsilon)$ such that the following inequalities hold:

$$\begin{aligned} \sum_{k=1}^{M-1} k \mu(A_k) &> 1 - \epsilon/3, \\ \epsilon/2 &> 2e^{-1} \sum_{k \geq M} e^{-k/2}. \end{aligned}$$

The hypothesis $\mu(\partial\mathcal{Q}) = 0$ implies that $\eta \mapsto \eta(P)$ is continuous at μ for each atom P of the partition \mathcal{Q} . We can find a neighborhood \mathcal{W} of μ such that the following two inequalities hold

$$\begin{aligned} \epsilon - \sum_{k < M} \sum_{P \in S_k} \mu(P) \log \mu(P) - \mu(X \setminus K) \log \mu(X \setminus K) \\ &> - \sum_{k < M} \sum_{P \in S_k} \eta(P) \log \eta(P) - \eta(X \setminus K) \log \eta(X \setminus K), \\ \sum_{k=1}^{M-1} k \eta(A_k) &> 1 - \epsilon/2. \end{aligned}$$

In particular $\sum_{k \geq M} k \eta(A_k) < \epsilon/2$. Define $S_M = \{n \geq M : x_n > e^{-n}\}$ and observe that $t^{1/2} \log(t^{-1}) \leq 2e^{-1}$, for every $t > 0$. Then by separating the following sum in terms of S_M and its complement we obtain

$$- \sum_{k \geq M} \eta(A_k) \log(\eta(A_k)) \leq \sum_{k \geq M} k \eta(A_k) + 2e^{-1} \sum_{k \geq M} e^{-k/2} < \epsilon.$$

Combining this inequalities and the estimate above we get that for each $\eta \in \mathcal{W}$

$$- \sum_{k \geq M} \sum_{P \in S_k} \eta(P) \log \eta(P) < \frac{1}{2} \epsilon d \log L + \frac{1}{2} \log(C/r^d) \epsilon + \epsilon.$$

Define $C(\epsilon) = \frac{1}{2} \epsilon d \log L + \frac{1}{2} \log(C/r^d) \epsilon + 2\epsilon$. Finally

$$H_\eta(\mathcal{Q}) \leq C(\epsilon) + H_\mu(\mathcal{Q}).$$

This gives the upper semicontinuity of $\eta \mapsto H_\eta(\mathcal{Q})$ at μ . For the lower semicontinuity we don't need to know anything else than $\mu(\partial\mathcal{Q}) = 0$, we leave the details to the reader. \square

Since for every $n \in \mathbb{N}$ the map $\eta \mapsto H_\eta(\mathcal{Q}^n)$ is continuous at μ and infimum of continuous functions is upper semicontinuous, we conclude that the map $\eta \mapsto h_\eta(\mathcal{Q})$ is upper semicontinuous at μ . This finishes the proof of the theorem. \square

We finish this section with a criterion that prevent escape of mass and a consequence. Recall that a continuous function $F : X \rightarrow Y$ is *proper* if the preimage of any compact subset of Y is compact.

Lemma 2. *Let X be a noncompact topological space and $F : X \rightarrow \mathbb{R}$ a positive proper function. Let $\{\mu_n\}_{n \geq 1}$ be a sequence of probability measures on X converging weakly to μ . Suppose that there exists n_0 and C such that for $n \geq n_0$ we have*

$$\int_X F(x) d\mu_n(x) \leq C.$$

Then μ is a probability measure.

Proof. Define $\Delta = 1 - \mu(X)$. We want to prove $\Delta = 0$. Let $K(m) = \{x \in X : f(x) \leq m\}$. For $n \geq n_0$ we have the inequality

$$m \mu_n(X \setminus K(m)) \leq \int_X F(x) d\mu_n(x) \leq C.$$

By definition of vague convergence we have

$$\lim_{n \rightarrow \infty} \mu_n(X \setminus K) = \Delta + \mu(X \setminus K),$$

for each compact subset K of X satisfying $\mu(\partial K) = 0$. In particular for most choices of m we get

$$\lim_{n \rightarrow \infty} m\mu_n(X \setminus K(m)) = m\Delta + m\mu(X \setminus K(m)).$$

If $\Delta \neq 0$ this leads to a contradiction because of the inequality above. \square

Let F be a positive continuous function on X . Define

$$\mathcal{M}_T^{F,D} = \{\mu \in \mathcal{M}_T(X) : \int F d\mu \leq D\},$$

and

$$P^{F,D}(f) = \sup_{\mu \in \mathcal{M}_T^{F,D}} (h_\mu(T) + \int f d\mu).$$

Proposition 2. *Let X be a noncompact topological space and $F : X \rightarrow \mathbb{R}$ a positive proper function. Then $\mathcal{M}_T^{F,D}$ is a compact subset of \mathcal{M}_T . Moreover assume that (X, T) satisfies the hypothesis of Theorem 6. Then for every ergodic $\mu \in \mathcal{M}_T^{F,D}$ we have the formula*

$$h_\mu(T) = \inf_{f \in C(X)} \{P^{F,D}(f) - \int f d\mu\}.$$

In particular the entropy map restricted to $\mathcal{M}_T^{F,D}$ is upper semicontinuous.

Proof. From Lemma 2 it is easy to obtain that $\mathcal{M}_T^{F,D}$ is compact for any positive proper function F . For the second part of the proposition we refer the reader to (19.10) in [DGK], since the proof is identical. \square

4. GEODESIC FLOW ON NONPOSITIVELY CURVED MANIFOLDS

In this section we prove Theorem B as an application of Theorem A and the proof of the main result in [RV]. From now on (M, g) will be a smooth complete Riemannian manifold of nonpositive curvature and sectional curvature bounded below. Let $g_t : T^1M \rightarrow T^1M$ be the geodesic flow on M and $\pi : T^1M \rightarrow M$ the canonical projection. If d_0 is the metric induces by the Riemannian metric g on M . Then the formula

$$d(v, w) = \max_{t \in [0,1]} d_0(\pi(g_t(v)), \pi(g_t(w)))$$

defines a metric in T^1M . Let (\widetilde{M}, g) be the universal cover of (M, g) . Because the curvature is bounded it follows easily that $(T^1\widetilde{M}, d)$ has bounded geometry. The following lemma will help to prove the simplified entropy formula for the geodesic flow.

Lemma 3. *Let (M, g) be a complete Riemannian manifold without focal points and bounded sectional curvature. Then $(T^1\widetilde{M}, d)$ has bounded geometry and $(T^1\widetilde{M}, d, g_1)$ has Property PC.*

Proof. As mentioned above, the bounded geometry condition follows from the bounded sectional curvature and the fact that the exponential map on \widetilde{M} is a diffeomorphism. Let $f_1(t)$ and $f_2(t)$ be two rays in \widetilde{M} such that $f_1(0) = f_2(0)$. Since by assumption M has not focal points we know that $d(f_1(t), f_2(t))$ is increasing in t . This implies that given two geodesic segments parametrized by arc length $g_1(t)$ and $g_2(t)$, then

$$d(g_1(t), g_2(t)) \leq d(g_1(t_0), g_2(t_0)) + d(g_1(t_1), g_2(t_1)),$$

for all $t \in (t_0, t_1)$. For a proof join $g_1(t_0)$ and $g_2(t_1)$ and use the increasing property above. This easily implies that the geodesic flow on (\widetilde{M}, g) has property PC (one need to take care of the distance 1 error used in the definition of d , but for segments of big enough length we can control it). \square

Proposition 3. *Let (M, g) be a Riemannian manifold with nonpositive sectional curvature and curvature bounded below. Let μ be an ergodic g_1 -invariant probability measure. Given $\delta \in (0, 1)$, there exist a positive number $r_0 = r_0(\delta)$ for which*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log N_\mu(n, r; \delta/2),$$

is well defined and independent of r , for any $r < r_0$. In particular this limit is equal to $h_\mu(g)$ for any ergodic g_1 -invariant probability measures μ .

Proof. We start with the following Lemma.

Lemma 4. *Assume (Y, ρ, S) has Property PC and (Y, ρ) has bounded geometry. Let r, r' be positive numbers. There exists a constant $L(r', r)$ such that $B_n(y, r)$ can be covered by at most $L(r, r')$ (n, r') -dynamical balls.*

Proof. Without lose of generality we can assume $r' < r$. For every $y \in Y$ we cover $B_n(y, r)$ and $B(S^n(y), r)$ by balls of radius $\epsilon(r')$ (using notation in Definition 2). By Definition 3 we can cover each of them by at most $C(r, \epsilon(r'))$ balls. Denote by $\{B_i\}$ and $\{B'_j\}$ the family of balls of radius $\epsilon(r')$ covering $B(y, r)$ and $B(S^n(y), r)$ respectively. The covering $\mathcal{U} = \{B_i \cap S^{-n}B'_j\}$ of $B_n(y, r)$ has at most $C(r, \epsilon(r'))^2$ elements. Using Property PC it easily follows that each element in \mathcal{U} is contained in a (n, r') -dynamical ball of Y . It follows that $B(y, r)$ can be covered by at most $C(r, \epsilon(r'))^2$ (n, r') -dynamical balls, and that this is independent of y . \square

Define $X = T^1M$ and $\pi_0 : X \rightarrow M$ be the canonical projection. Let K be a compact subset of X satisfying $\mu(K) > 1 - \delta/2$, and $\mu(\partial K) = 0$. Pick \tilde{K}_0 a compact fundamental domain for $K_0 = \pi_0(K)$ on the universal cover \widetilde{M} of M . For a point $x \in K_0$ we denote by \tilde{x} to its lift to \tilde{K}_0 . Because \tilde{K}_0 is compact and $\Gamma = \pi_1(M)$ acts discretely on \widetilde{M} , then

$$D = \inf_{\gamma \in \Gamma, w \in \tilde{K}_0} d(\gamma w, w),$$

is a positive number. Because \widetilde{M} has bounded curvature, there exist constants $C(D)$ and $L(D)$ such the following hold. Given a, b and c points in \widetilde{M} such that $d(c, a) = d(c, b) = L$ and $d(a, b) > D$, then $d(\bar{a}, \bar{b}) > C(D)$, where \bar{a} (resp. \bar{b}) is the midpoint of the segment $[c, b]$ (resp. $[c, a]$) and $L > L(D)$. Choose a positive number r satisfying $r < \inf\{C(D)/4, D/2\}$. This choice of r has the following consequences. Let $z \in T^1\tilde{K}_0$ such that $g_L(z) \in \gamma T^1\tilde{K}_0$, for $L > L(D)$ and $\gamma \in \Gamma$. Then the (L, r) -dynamical ball centered at z in $T^1\widetilde{M}$ is in bijection with $B_L(d\pi(z), r)$, the dynamical ball in X . More concretely, a dynamical ball in T^1M with sufficiently small radius will lift to a unique dynamical ball in $T^1\widetilde{M}$, this is a priori not clear to occur because our geodesic might enter in regions where the injectivity radius is arbitrarily small. In Figure 1 the tubes represent r -neighborhoods of the geodesic emanating from z and its translations by Γ . Since for $x, y \in M$ we have $d(x, y) = \inf_{\gamma \in \Gamma} d(\tilde{x}, \gamma \tilde{y})$, points in $T^1\widetilde{M}$ that remain close to the projection to T^1M of the red trajectory are precisely those that remain close to the union of all possible translates of the

red trajectory (the dashed trajectories in Fig.1). The existence of a geodesic as the blue line would be an obstruction for the correspondence between dynamical balls in T^1M and on $T^1\widetilde{M}$, we ruled out this case for sufficiently large L and small enough r . If the injectivity radius of M was bounded below this discussion can be omitted by taking r smaller than the injectivity radius.

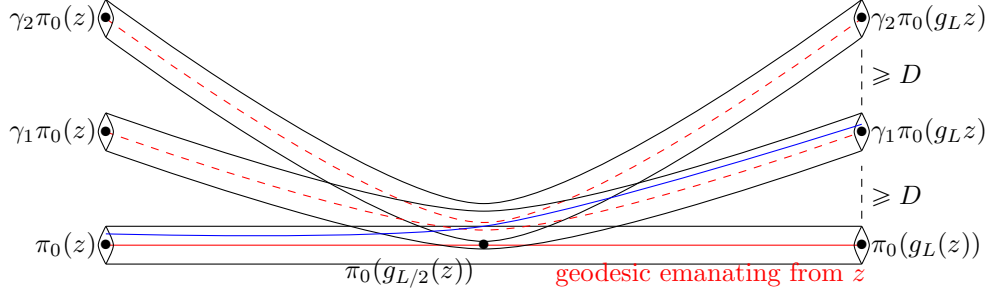


Figure 1: Thrice-puncture sphere with one cusp

For now on we assume $n > L(D)$. Choose $r' \in (0, r)$ and let y be the lift of $x \in K$ to $T^1\widetilde{M}$ based at \tilde{K}_0 . By Lemma 3 and Lemma 4 we can cover $B_n(y, r)$ by at most $L(r, r')$ (n, r') -dynamical balls in $T^1\widetilde{M}$. By the discussion above we conclude that $B_n(x, r)$ can be covered by $L(r', r)$ (n, r') -dynamical balls. We summarize this by saying that for any $A \subset K$ we have $N(A, n, r') \leq L(r, r')N(A, n, r)$. Let K_n be a compact subset of X with $\mu(K_n) > 1 - \delta/2$ such that $N(K_n, n, r) = N_\mu(n, r; \delta/2)$. Define $C_n = K_n \cap K$ and observe that $\mu(C_n) \geq 1 - \delta$. Then

$$\begin{aligned} N_\mu(n, r'; \delta) &\leq N(C_n, n, r') \leq L(r, r')N(C_n, n, r) \\ &\leq L(r, r')N(K_n, n, r) = L(r, r')N_\mu(n, r; \delta/2). \end{aligned}$$

This implies that for every $r' \in (0, r)$, we have

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log N_\mu(n, r'; \delta) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log N_\mu(n, r; \delta/2).$$

In particular

$$h_\mu(g) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log N_\mu(n, r; \delta/2).$$

The RHS is decreasing as a function of r , then we also have

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log N_\mu(n, r; \delta/2) \leq h_\mu(g),$$

which implies the equality. This discussion applies to \limsup as well, we conclude the limit exists and it is independent of r , for r sufficiently small. \square

It worth mentioning that without any modification the same arguments hold for the geodesic flow on a complete Riemannian manifold without focal points and bounded sectional curvature. Combining Proposition 3 and Theorem 6 we obtain the following corollary.

Corollary 1. *Let (X, g) be a complete Riemannian manifold without focal points and bounded sectional curvature. Let $\{\mu_n\}_{n \geq 0}$ be a sequence of g_1 -invariant probability measures converging to an ergodic g_1 -invariant probability measure μ . Then*

$$\limsup_{n \rightarrow \infty} h_{\mu_n}(g_1) \leq h_{\mu}(g_1).$$

We remark that (X, g) does not need to be compact. It is not hard to see that the geodesic flow on a compact nonpositively curved manifolds is h -expansive, therefore in the compact case the upper semicontinuity of the entropy map follows from [Bo2]. It worth mentioning that Corollary 1 has been recently obtained in [RV] if (X, g) is a pinched negatively curved geometrically finite manifold by different methods. Using results of Bowditch [B] on the structure of the nonwandering set of the geodesic flow we can say much more in that case. Recall that if X is geometrically finite, then its fundamental group has a finite number of maximal parabolic subgroups (up to conjugation). The maximal parabolic subgroups (up to conjugation) are in 1-1 correspondence with the cusps of X . For a discrete subgroup Γ of $\text{Iso}(\tilde{X})$ we can define

$$\delta_{\Gamma} = \limsup_{n \rightarrow \infty} \frac{1}{n} \log(\#\{\gamma \in \Gamma : d(x, \gamma x) < n\}).$$

This number is called the *critical exponent* of Γ . Define

$$\bar{\delta}_{\mathcal{P}} = \max_{\mathcal{P}} \delta_{\mathcal{P}},$$

where the maximum runs over the maximal parabolic subgroups of $\pi_1(X)$. The main result of [RV] is the following.

Theorem 7. *Let (X, g) be a geometrically finite Riemannian manifold with pinched negative sectional curvature. Let (μ_n) be a sequence of ergodic (g_t) -invariant probability measures on T^1X converging to μ in the vague topology. Then*

$$\limsup_{n \rightarrow \infty} h_{\mu_n}(g_1) \leq \|\mu\| h_{\frac{\mu}{\|\mu\|}}(g_1) + (1 - \|\mu\|) \bar{\delta}_{\mathcal{P}}.$$

The importance of the geometrically finite assumption is that the ends of the nonwandering set of the geodesic flow go through the cusps of T^1X and that each cusp is standard, i.e. the quotient of a horoball by the action of some parabolic subgroup of $\pi_1(X)$. We remark that any finite volume negatively curved manifold is geometrically finite. If X is just nonpositively curved we have very little control over the ends of X , even if X is finite volume. The ends of a finite volume nonpositively curved manifolds can contain a lot of topology, for example see Section 5 in [G]. This is an obstruction to employ our strategy. We now recall the class of manifolds on which we will focus on.

Definition 5. *Let \tilde{X} be a complete simply connected Riemannian manifold of non-positive sectional curvature. We say that \tilde{X} satisfies the Visibility axiom if for each $x \in \tilde{X}$ and $\epsilon > 0$, there exists $R = R(p, \epsilon)$ such that $\angle_p(\gamma(a), \gamma(b)) \leq \epsilon$ for every geodesic segment $\gamma : [a, b] \rightarrow \tilde{X}$ with $d(p, \gamma) \geq R$.*

The following theorem is the reason of the hypothesis in Theorem B. In the negative curvature case this theorem follows by the work of Gromov and Heintze. As mentioned above, if we remove the Visibility axiom hypothesis there are many examples of finite volume nonpositively curved manifolds with non-standard ends.

Theorem 8 (P. Eberlein). *Let X be a complete nonpositively curved manifold with curvature bounded below and finite volume. Moreover assume its universal cover \tilde{X} satisfies the Visibility axiom. Then X has finitely many ends and each of them is a standard cusp, i.e. isometric to the quotient of a horoball by a parabolic subgroup \mathcal{P} of Γ . In particular the cusps of X are π_1 -injective.*

Before starting the proof of the second part of Theorem B we collect some facts about nonpositively curved manifolds that we will use. From now on we will always assume that the hypothesis of Theorem 8 hold. Fix a reference point $o \in \tilde{X}$. We denote by $\partial\tilde{X}$ the boundary at infinity of \tilde{X} . For $x, y \in \tilde{X}$ and $\xi \in \partial_\infty \tilde{X}$ we define the *Busemann function* as

$$b_\xi(x, y) = \lim_{t \rightarrow \infty} d(x, \xi_t) - d(y, \xi_t),$$

where $t \mapsto \xi_t$ is any geodesic ray ending at ξ . For every $\xi \in \partial_\infty$ and $s > 0$, denote by $B_\xi(s)$ the horoball centered at ξ of height s relative to o , that is

$$B_\xi(s) = \{y \in \tilde{M} : b_\xi(o, y) \geq s\},$$

where $b_\xi(o, \cdot)$ is the Busemann function at ξ relative to o . By Theorem 8 we know that there exists a maximal finite collection $\{\xi_i\}_{i=1}^{N_p}$ of non equivalent parabolic fixed points in $\partial_\infty \tilde{X}$, each of them corresponding to a maximal parabolic subgroup \mathcal{P}_i . Moreover, there exists $s_0 > 0$, such that for every $s \geq s_0$ the collection $\{B_{\xi_i}(s)/\mathcal{P}_i\}_{i=1}^{N_p}$ are disjoint cusp neighbourhoods for X . For every $s \geq s_0$ define

$$X_{>s} = \bigcup_{i=1}^{N_p} T^1 H_{\xi_i}(s)/\mathcal{P}_i \quad \text{and} \quad X_{\leq s} = X \setminus X_{>s}.$$

As in the geometrically finite case, the following quantity will be important for us:

$$\bar{\delta}_{\mathcal{P}} = \max_{\mathcal{P}} \delta_{\mathcal{P}},$$

where the maximum runs over the maximal parabolic subgroups of $\pi_1(X)$. The following lemma corresponds to Lemma 3.1 in [RV] and is the main reason why we need X to have standard cusps.

Lemma 5. *Let $s > s_0$. There exists $l_s \in \mathbb{N}$ such that whenever $v \in X_{\leq s_0}$ satisfies $gv, \dots, g^k v \in X_{\leq s} \cap X_{>s_0}$ and $g^{k+1}v \in X_{\leq s_0}$, then necessarily we have $k \leq l_s$.*

Proof. It is enough to prove the statement for orbits in the same cusp component of X . Observe that by the finite volume hypothesis the set $X_{\leq s} \cap X_{>s_0}$ is relatively compact in X , denote by A_s one of its connected components. Suppose by contradiction that there are arbitrarily long orbit in A_s . Denote by $O_n \subset A_s$ a g_1 -orbit of length at least n and let η_n the equidistribution measure supported in O_n . It is an easy exercise in Ergodic theory that the sequence of probability measures η_n converges to a g_1 -invariant probability measure η supported \bar{A}_s . By construction \bar{A}_s embeds in a manifold of the form $T^1 \tilde{X}/\mathcal{P}$. The geodesic flow on X/\mathcal{P} is extremely simple, there are no compact invariant subset for g_1 (here we are using again that \tilde{X} satisfies the visibility axiom). This contradicts the existence of η and therefore finishes the proof. \square

Lemma 6. *There exists a compact subset $K \subset T^1 X$ such that $\mu(K) > 0$ for all g_1 -invariant probability measure.*

The proof is identical to the one in Lemma 5, we leave the details to the reader. We say that an interval $[a, a + b]$ is an excursion of $Y \subset X$ into $X_{>s}$ if

$$\begin{aligned} g^{a-1}Y, g^{a+b}Y &\subset X_{\leq s_0}, \\ g^aY, \dots, g^{a+b-1}Y &\subset X_{\geq s_0} \end{aligned}$$

Let $n \geq 1$. We denote by $|E_{s,n}(Y)|$ the sum of the length of all the excursions of Y into $X_{>s}$ for times in $[0, n]$. We also denote by $m_{s,n}(Y)$ the number of excursions of Y into $X_{>s}$. The following lemma follows from Property PC and Lemma 5, for a proof we refer the reader to Proposition 3.3 in [RV].

Lemma 7. *Let $r > 0$ and $s > s_0$. Suppose that $\beta = \{X_{>s}, X_{\leq s} \cap X_{>s_0}, Q_1, \dots, Q_b\}$ is a finite partition of X such that $\text{diam}(g^j(Q_k)) < \epsilon(r)$ for every $0 \leq j \leq l_s$ and every $1 \leq k \leq b$. Then there exists a constant $C_{r,s_0} \geq 1$ such that for each $n \geq 1$ and $Q \in \beta_0^n$ with $Q \subset X_{\leq s_0}$, the set Q can be covered by*

$$C_{r,s_0}^{m_{s,n}(Q)} e^{\bar{\delta}_{\mathcal{P}} |E_{s,n}(Q)|}$$

(n, r) -dynamical balls.

We remark that $\epsilon(r)$ is the quantity that appears in the definition of Property PC for the action of the geodesic flow on $T^1\tilde{X}$. The following Proposition follows directly from Lemma 7 and Lemma 6 and as pointed out in [RV], it implies Theorem 9 below (once the simplified entropy formula is verified).

Proposition 4. *Let β be a partition of X as in Lemma 7. Then for every ergodic g_1 -invariant probability measure μ on X , the following inequality holds*

$$h_\mu(g) \leq h_\mu(g, \beta) + \mu(X_{>s_0}) \bar{\delta}_{\mathcal{P}} + \frac{1}{s - s_0} \log C_{r,s_0}.$$

Theorem 9. *Let X be a complete nonpositively curved manifold with curvature bounded below and finite volume. Assume its universal covering satisfies the visibility axiom. Let (μ_n) be a sequence of ergodic (g_1) -invariant probability measures on T^1X converging to μ in the vague topology. Then*

$$\limsup_{n \rightarrow \infty} h_{\mu_n}(g_1) \leq \|\mu\| h_{\frac{\mu}{\|\mu\|}}(g_1) + (1 - \|\mu\|) \bar{\delta}_{\mathcal{P}}.$$

This result together with Corollary 1 proves Theorem B. We now proceed to state a generalization of Theorem B. We start with some definitions. An open manifold Y is called *topologically tame* if it is homeomorphic to the interior of a compact manifold with boundary. A topologically tame manifold is said to have *incompressible boundary* if the inclusion of each boundary component induces an injection at the level of fundamental groups. In particular each boundary component B of Y induces a subgroup $E_B \cong \pi_1(B)$ of $\pi_1(Y)$. We endow Y with a Riemannian metric g , as usual this induces a distance function d . As before, one can define the critical exponent of any subgroup of $\pi_1(Y)$ by its action at the universal cover of Y . We denote by δ_B the critical exponent of E_B . We define *critical exponent at infinity* of Y as

$$\delta_\infty = \sup_B \delta_B,$$

where the supremum runs over the boundary components of Y . Given a continuous transformation T on Y , we say that the dynamics generated by T is *weakly bounded* if there exists a compact set K intersecting every T -invariant subset of Y . In other

words the T -invariant subsets of Y can not escape to infinity. With all this notation we can now state a generalization of Theorem 9.

Theorem 10. *Let (Y, g) be a topologically tame complete Riemannian manifold with incompressible boundary and without focal points. Assume that the sectional curvature is bounded and that the dynamical system (T^1Y, d, g_1) is weakly bounded. Let (μ_n) be a sequence of ergodic g_1 -invariant probability measures on Y converging to μ in the vague topology. Then*

$$\limsup_{n \rightarrow \infty} h_{\mu_n}(g) \leq \|\mu\| h_{\frac{\mu}{\|\mu\|}}(g) + (1 - \|\mu\|) \delta_\infty.$$

In particular, if $\delta_\infty < h_{\text{top}}(g)$, then there exists a measure of maximal entropy.

Since we can check that Lemma 5 and Lemma 6 are satisfied, we can follow the proof of Theorem 9 to get Theorem 10. We hope that the hypothesis of Theorem 10 will be verified for a much larger class of nonpositively curved manifolds.

5. SYMBOLIC DYNAMICS

We start by recalling some facts about symbolic dynamics. Let S be an alphabet with at most countable many symbols and M be a $S \times S$ matrix with entries 0 or 1. We define the symbolic space associated to M as

$$\Sigma = \{x = (x_0, x_1, \dots) \in S^{\mathbb{N}} : M(x_i, x_{i+1}) = 1\}.$$

We endow S with the discrete topology and $S^{\mathbb{N}}$ with the product topology. On Σ we consider the induced topology given by the natural inclusion $\Sigma \subset S^{\mathbb{N}}$. Let $\sigma : \Sigma \rightarrow \Sigma$ be the shift map, i.e. $\sigma(x_0, x_1, x_2, \dots) = (x_1, x_2, \dots)$. Clearly σ is a continuous map. We say that (Σ, σ) is a *countable topological Markov shift* if it comes from this construction. We denote by $[a_0, \dots, a_{N-1}]$ the set of sequences $x = (x_0, \dots) \in \Sigma$, where $x_i = a_i$ for $0 \leq i \leq N-1$. We say that $[a_0, \dots, a_{N-1}]$ is a cylinder of length N or a N -cylinder. Fix $\theta \in (0, 1)$ and define a metric $d = d_\theta$ on Σ by declaring $d(x, y) = 1$ if $x_0 \neq y_0$, and $d(x, y) = \theta^k$ if k is the length of the biggest cylinder containing x and y . We say that μ is a Gibbs measure if there exist real numbers P, G , and a continuous function $F : \Sigma \rightarrow \mathbb{R}$ such that

$$G^{-1} \exp(-N(P - S_N F(x))) \leq \mu([a_0, \dots, a_{N-1}]) \leq G \exp(-N(P - S_N F(x))),$$

for every $x \in [a_0, \dots, a_{N-1}]$, where $S_N F(x) = \frac{1}{n} \sum_{n=0}^{N-1} F(\sigma^n x)$. The following result describes very precisely how Gibbs measures appear in the countable alphabet setting. Since our proofs do not require most of the notation used in the statement of this result, we will take it as a black box. We refer the reader to [Sa2] for precise definitions and its proof.

Theorem 11. *Let (Σ, σ) be a topologically mixing countable Markov shift. A Walters function F admits a Gibbs measure μ if and only if*

- (1) Σ satisfies the BIP property,
- (2) $P_G(F) < \infty$ and $\text{var}_1(F) < \infty$.

In this case F is positive recurrent, μ is the Ruelle-Perron-Frobenius measure of F and P is equal to $P_G(F)$. If $\sup F < \infty$ and F has summable variations then μ is ergodic and unique.

In other words, if we assume that Σ is a topologically mixing countable Markov shift satisfying the BIP property, and the potential F is sufficiently regular, then there exists a unique Gibbs measure associated to F . In particular if Σ is a topologically mixing countable Markov shift satisfying the BIP property we have plenty of ergodic Gibbs measures.

Theorem 12. *Let (Σ, σ) be a countable Markov shift and μ an ergodic Gibbs measure which is an equilibrium state for its potential. Then for every sequence $\{\mu_n\}_{n \geq 1}$ of ergodic σ -invariant probability measures converging to μ we have*

$$\limsup_{n \rightarrow \infty} h_{\mu_n}(\sigma) \leq h_\mu(\sigma).$$

Proof. It is an easy exercise to modify the proof of Lemma 2.5 in [RV] to the symbolic setting, in particular we have the following result.

Lemma 8. *Assume (Σ, d, σ) admits an ergodic Gibbs measure. Then for any σ -invariant probability measure μ , the expression*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log N_\mu(n, r; \delta),$$

is well defined and it is independent of r .

We will also need the equality in Katok's entropy formula for an ergodic Gibbs measure. Since the proof is similar to the one of Theorem 12 we leave the details to the reader.

Lemma 9. *Let μ be an ergodic Gibbs measure on (Σ, d, σ) satisfying*

$$P = h_\mu(\sigma) + \int F d\mu.$$

Then

$$h_\mu(\sigma) = \lim_{n \rightarrow \infty} \frac{1}{n} \log N_\mu(n, r; \delta).$$

We can finally start the proof of Theorem 12. Define

$$A_n = \{x \in \Sigma : |S_N F(x) - \int F d\mu| < \epsilon, \text{ for } N \geq n\}.$$

Observe that the continuity of F implies that each A_n is open. By Birkhoff ergodic theorem we have $\lim_{n \rightarrow \infty} \mu(A_n) = 1$. Choose n big enough such that $\mu(A_n) > 1 - \delta/2$. Define

$$B_n^N = \bigcup_{x \in A_n} B_N(x, \theta^k),$$

where $B_N(x, r)$ is the (open) (n, r) -dynamical ball centered at x . Observe that $B_N(x, \theta^k) = [x_0, \dots, x_{N+k}]$. Choose a neighborhood \mathcal{W} of μ such that for every $\eta \in \mathcal{W}$ we have $\eta(A_n) > 1 - \delta$. Using the lower bound for the measure of a dynamical balls in B_n^N we get

$$N_\eta(N, \theta^k; \delta) \leq N(B_n^N, N, \theta^k) \leq G \exp((N + k + 1)(P - \int F d\mu + \epsilon)).$$

Let K be a subset satisfyig $N_\mu(N, \theta^k, \delta/2) = N(K, N, \theta^k)$ and $\mu(K) > 1 - \delta/2$. Then $\mu(K \cap A_n) > 1 - \delta$, which together with the upper bound for the measure of

dynamical balls containing points in A_n imply

$$\begin{aligned} G(1 - \delta) \exp((N + k + 1)(P - \int F d\mu - \epsilon)) &< N(K \cap A_n, N, \theta^k) \\ &\leq N(K, N, \theta^k) = N_\mu(N, \theta^k; \delta/2). \end{aligned}$$

We can combine this inequalities and get

$$N_\eta(N, \theta^k; \delta) \leq N_\mu(N, \theta^k; \delta/2) \exp(2\epsilon(N + k + 1))(1 - \delta)^{-1}.$$

Finally using Theorem 5 and Lemma 8 we get $h_\eta(\sigma) \leq h_\mu(\sigma) + 2\epsilon$, for each ergodic measure $\eta \in \mathcal{W}$. \square

Let (X, T) be a dynamical system and $\tau : X \rightarrow \mathbb{R}^+$ a continuous function. Define the space $Y_\tau = X \times \mathbb{R}_{\geq 0} / \sim$, where $(x, \tau(x)) \sim (Tx, 0)$. On Y_τ we can define the semiflow $\Phi = \{\Phi_t\}_{t \geq 0}$ given by $\Phi_t([(x, s)]) = [(x, t + s)]$. The *suspension flow* of (X, T) with roof function τ is the semiflow (Y_τ, Φ) . Denote by \mathcal{M}_Φ the space of Φ -invariant probability measures on Y_τ and by \mathcal{M}_T the space of T -invariant probability measures on X . It follows from a classical result by Ambrose and Kakutani [AK] that every measure $\nu \in \mathcal{M}_\Phi$ can be written as

$$\nu = \frac{(\mu \times m)|_{Y_\tau}}{(\mu \times m)(Y_\tau)},$$

where μ is a T -invariant measure and m denotes the one dimensional Lebesgue measure. In the opposite direction, if μ is a T -invariant probability measure on X satisfying $(\mu \times m)(Y_\tau) = \int \tau d\mu < \infty$, then we can define a Φ -invariant probability measure $A(\mu)$ given by the formula

$$A(\mu) = \frac{(\mu \times m)|_{Y_\tau}}{(\mu \times m)(Y_\tau)}.$$

Define $\mathcal{M}_T(\tau) = \{\mu \in \mathcal{M}_T : \int \tau d\mu < \infty\}$. We say that τ is bounded away from zero if $\tau(x) \geq c > 0$ for all $x \in X$. It follows from results in [AK] that if τ is bounded away from zero, then the map $A : \mathcal{M}_T(\tau) \rightarrow \mathcal{M}_\Phi$ is a bijection.

Proposition 5. *Let (Σ, σ) be a countable Markov shift and μ an ergodic Gibbs measure which is an equilibrium state for its potential. Let $\tau : X \rightarrow \mathbb{R}^+$ be a bounded away from zero continuous function such that the suspension flow (Y_τ, Φ) has finite topological entropy. Consider a sequence $\{\nu_n\}$ of ergodic Φ -invariant probability measures on Y_τ converging to $\nu = T(\mu)$. Then*

$$\limsup_{n \rightarrow \infty} h_{\nu_n}(\Phi) \leq h_\nu(\Phi).$$

Proof. Denote by $\mu_n \in \mathcal{M}_\sigma(\tau)$ the measure such that $\nu_n = A(\mu_n)$. It is easy to see that if ν_n is ergodic, then the same holds for μ_n . Observe that $\int_K \tau d\mu = \lim_{n \rightarrow \infty} \int_K \tau d\mu_n \leq \liminf_{n \rightarrow \infty} \int \tau d\mu_n$. Then

$$\int \tau d\mu \leq \liminf_{n \rightarrow \infty} \int_K \tau d\mu_n.$$

Abramov's formula [Ab] says that $h_{\nu_n}(\Phi) = \frac{h_{\mu_n}(\sigma)}{\int \tau d\mu_n}$. Combining this observations with the semicontinuity result obtained in Theorem 12 finish the proof. \square

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